

RIGHT-ANGLED ARTIN GROUP BOUNDARIES

MICHAEL BEN-ZVI AND ROBERT KROPHOLLER

ABSTRACT. In all known examples of a $\text{CAT}(0)$ group acting on $\text{CAT}(0)$ spaces with non-homeomorphic $\text{CAT}(0)$ visual boundaries, the boundaries are each not path connected. In this paper, we show this does not have to be the case by providing examples of right-angled Artin groups which exhibit non-unique $\text{CAT}(0)$ boundaries where all of the boundaries are arbitrarily connected. We also prove a combination theorem for certain amalgams of $\text{CAT}(0)$ groups to act on spaces with non-path connected visual boundaries. We apply this theorem to some right-angled Artin groups.

1. INTRODUCTION

When Gromov introduced hyperbolic groups [Gro87], he showed that their boundaries are well-defined in the sense that if Γ is hyperbolic and acts geometrically on spaces X and Y , then ∂X is homeomorphic to ∂Y . He asked if the same is true for $\text{CAT}(0)$ groups. Croke and Kleiner answered this question in the negative using the fundamental group of the surface amalgam in Figure 1 [CK00]. Throughout we denote this group CK . Changing the angle of intersection between the curves b and c from $\pi/2$ to anything else changes the universal cover in a way which makes the resulting boundaries non-homeomorphic. Later, Wilson showed that for any pair of angles, the corresponding universal covers have non-homeomorphic boundary [Wil05]. Thus CK admits uncountably many visual boundaries. Croke and Kleiner extended their results to a larger class of groups [CK02] and Mooney further generalized this work, providing even more examples of $\text{CAT}(0)$ groups with non-unique boundary [Moo10].

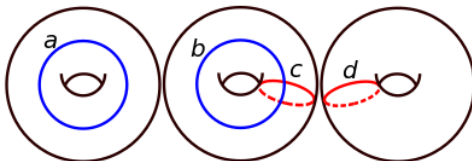


FIGURE 1. Three tori with the curve a identified with b and the curve c identified with d .

The boundaries for every one of these groups has one thing in common: it is not path connected. This leads to the following questions

Question 1.1. Suppose G acts geometrically on a CAT(0) spaces X and Y .

- If ∂X is path connected, are ∂X and ∂Y homeomorphic?
- If ∂X is n -connected, are ∂X and ∂Y homeomorphic?

In this paper, we show the answer to these questions is ‘no.’

Theorem 4.1. *For each n , there is a group G_n and CAT(0) spaces X_n and Y_n admitting geometric group actions by G_n with the following properties:*

- ∂X_n and ∂Y_n are n -connected
- ∂X_n and ∂Y_n are not homeomorphic

We do this by considering $G_n = CK \times \mathbb{Z}^{n+1}$. Once again the choice of angle between the curves b and c produces spaces with non-homeomorphic boundaries. One should note that due to the central \mathbb{Z}^{n+1} the boundary of any CAT(0) space upon which G_n acts geometrically has boundary $S^n * Z$, for some space Z , and so is n -connected. We ask the following question.

Question 1.2. Does there exist group G which acts on two CAT(0) spaces X, Y geometrically and ∂X is n -connected but ∂Y is not?

Another point of interest arising from studying CK is trying to understand when the phenomenon of the boundary not being path connected occurs. For hyperbolic groups path connectivity is equivalent to one-ended [BM91, Bow98, Swa96]. This is also true for groups which are CAT(0) and hyperbolic relative to free Abelian subgroups [Ben19]. In this paper we give a combination theorem which gives conditions on a space X together with a geometric action of $A *_C B$ to have path disconnected boundary. Namely, we prove the following:

Theorem 3.2. *Let $G = A *_C B$ be a CAT(0) group acting geometrically on a CAT(0) space X . Suppose that there is a subspace X_C with a geometric action of C which separates X . Suppose further that A acts geometrically on a subspace X_A satisfying the following:*

- (1) X_A has a connected block decomposition,
- (2) ∂X_A is not path connected, and
- (3) $\Lambda(C) \subset \text{Nex}(X_A)$

then ∂X is not path connected.

See Section 2.2 for relevant definitions.

The last requirement is needed in light of the example $F_2 *_\mathbb{Z} F_2$ giving a surface group. Thus it is not enough to assume that ∂X_A is not path connected.

Throughout the paper the example to have in mind is when $A = CK$ and $C = \langle a, d \rangle$. In this setting, CK acts naturally on its Salvetti complex and this space does not have a path connected visual boundary. If we then amalgamate over the visual F_2 subgroup, then the resulting group acts on a space with a non-path connected visual boundary.

The major application of this theorem comes in looking at right-angled Artin groups (RAAGs). Let A_Γ be the RAAG with defining graph Γ . Then we have sufficient conditions showing when S_Γ , the universal cover of the Salvetti complex of A_Γ , has a non-path connected visual boundary. Let P_4 be the defining graph for CK with vertex set $V(P_4) = \{a, b, c, d\}$ in the usual ordering.

Theorem 3.10. *Let A_Γ be a RAAG admitting a graph of groups as in Figure 2 where H_i is a proper parabolic subgroup of CK . Then ∂S_Γ is not path connected.*

One class of RAAGs which fits into the hypotheses of this corollary are those whose defining graph is an n -cycle for $n \geq 5$, which was the original motivation for proving Theorem 3.2. This fits into the following conjecture of Mihalik:

Conjecture 1.3. ∂S_Γ is path connected if and only if Γ is a join.

We can also study the groups from [Moo10]. These groups are of the form $(G_- \times \mathbb{Z}^n) *_{\mathbb{Z}^n} (\mathbb{Z}^n \times \mathbb{Z}^m) *_{\mathbb{Z}^m} (\mathbb{Z}^m \times G_+)$ where G_-, G_+ are infinite CAT(0) groups. As discussed above these groups are shown to have non-unique CAT(0) boundary [Moo10]. For certain of these groups we can show that the boundary is not path connected.

Theorem 3.11. *Let G be of the form $(G_- \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times G_+)$ or $(G_- \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}^2) *_{\mathbb{Z}^2} (\mathbb{Z}^2 \times G_+)$, where G_- and G_+ are CAT(0) groups. Then G acts on a CAT(0) X and ∂X is not path connected.*

We conjecture that this result holds more generally.

Conjecture 1.4. Let $G = (G_- \times \mathbb{Z}^n) *_{\mathbb{Z}^n} (\mathbb{Z}^n \times \mathbb{Z}^m) *_{\mathbb{Z}^m} (\mathbb{Z}^m \times G_+)$ where G_-, G_+ are infinite CAT(0) groups. Then G act on a CAT(0) space X such that ∂X is not path connected.

Outline. In Section 2, we provide the necessary background information on CAT(0) boundaries, block decompositions, and right-angled Artin groups. Section 3 is devoted to proving Theorem 3.2. We do this by showing if a closed set C separates closed sets A from B in X , then ∂C separates ∂A and ∂B in ∂X . We show that the hypotheses for Theorem 3.2 force this type of separation. In Section 3.1, we apply Theorem 3.2 to RAAGs and groups coming from [Moo10]. Lastly, in Section 4, we prove Theorem 4.1.

Acknowledgements. The authors thank Mike Mihalik and Kim Ruane for bringing the question of path-connectedness of boundaries to our attention.

2. PRELIMINARIES

2.1. CAT(0) spaces, groups, and boundaries. For a thorough introduction to CAT(0) spaces and groups, see [BH99]. Throughout this section, let X be a proper, complete CAT(0) space.

Definition 2.1 (Visual boundary and the cone topology). Fix a basepoint $q \in X$. The set $\partial_q X$ consists of geodesic rays based at q . Let $\overline{B}(q, r)$ be the closed ball of radius r about q . Let $\pi_r : X \rightarrow \overline{B}(q, r)$ be the projection map. Define the following sets in $\overline{X} = X \cup \partial_q X$:

$$U(c, r, D) := \{x \in \overline{X} : d(x, q) > r, d(\pi_r(x), c(r)) < D\}$$

where c is a geodesic segment or ray. Sets of this form along with balls in X make up a neighborhood basis for \overline{X} . This is called the *cone topology*.

Restricting the cone topology to $\partial_q X$, the neighborhood basis can be restated as follows:

$$U(c, r, D) = \{c' \in \partial_q X : d(c(r), c'(r)) < D\}$$

for any choice of $c \in \partial_q X$, $D > 0$, $r > 0$.

Proposition 2.2 ([BH99, II.8]). *For any two $q, q' \in X$, $\partial_q X$ and $\partial_{q'} X$ are homeomorphic. Because of this, we denote ∂X to be the visual boundary with the cone topology.*

Definition 2.3 (\mathcal{Z} -set). A closed subset Z in a compact absolute neighborhood retracts (ANR) Y is a \mathcal{Z} -set if the one of following equivalent conditions holds:

- (1) For every open set $U \subset Y$, $U - Z \hookrightarrow U$ is a homotopy equivalence.
- (2) For every closed $A \subset Z$, there is a homotopy $H : Y \times [0, 1] \rightarrow Y$ such that H_0 is the identity, $H_t|_A$ is the inclusion map, and $H_t(Y - A) \subset Y - Z$ for all $t > 0$.

It is known that \overline{X} is an (ANR) [Gui14], \overline{X} and ∂X are compact [BH99], and ∂X is a \mathcal{Z} -set [Bes96] in \overline{X} . The second point of the definition will be used in the proof of Lemma 3.1.

Definition 2.4 (n -connected). A topological space Z is n -connected for $n \geq 1$ if Z is non-empty, path connected, and $\pi_i(Z) = 0$ for $1 \leq i \leq n$. A space is 0-connected if it is non-empty and path connected.

2.2. Blocks and Itineraries.

Definition 2.5 ([Moo10, Defintion 3.1]). A block decomposition \mathcal{B} of a CAT(0) space X is a collection of closed, convex sets call *blocks* such that

- (1) $X = \bigcup_{B \in \mathcal{B}} B$,
- (2) each block intersects at least two other blocks,
- (3) Parity condition: every block has a (+) or (−) parity such that two blocks intersect only if they have opposite parity,
- (4) ϵ -condition: there is an $\epsilon > 0$ such that two blocks intersect iff their ϵ -neighborhoods intersect.

We say a block decomposition is *connected* if $\bigcup_{B \in \mathcal{B}} \partial B$ is path connected. When \mathcal{B} is a connected block decomposition, we shall refer to the path

component of ∂X containing a block boundary (equivalently all the block boundaries) as the *nexus* of X and will denote it $\text{Nex}(X)$.

A *wall* is a non-trivial intersection of blocks and the set of walls is denoted \mathcal{W} . A set C *separates* A and B if every path from A to B passes through C . When X has a block decomposition, if $B \cap B' = W$ for some $W \in \mathcal{W}$, then W separates B and B' .

The *nerve* of a block decomposition, denoted $\text{Nerve}(\mathcal{B})$, is a graph which records (non-trivial) block intersections. There is a vertex for each block and an edge if two blocks intersect. Following results from [CK00] and [Moo10], we get that the nerve is always a tree.

Lemma 2.6. *Nerve(\mathcal{B}) is a tree.*

Since $\text{Nerve}(\mathcal{B})$ is a tree, we will denote it $\mathcal{T}_{\mathcal{B}}$ or \mathcal{T} when \mathcal{B} is understood.

Fix a basepoint $x_0 \in X$ which is not in any wall. We say a geodesic ray $r: [0, \infty) \rightarrow X$ based at x_0 *enters* a block B if for some time $t > 0$, $r(t) \in B$ and $r(t) \notin B'$ for any other block B' . Associated to each ray is a *itinerary*, denoted $\text{Itin}_{x_0}(r)$, which is the sequence of blocks r enters. When the basepoint is understood we will denote this $\text{Itin}(r)$. If r represent the point $\alpha \in \partial X$, then we will write $\text{Itin}(\alpha)$ to denote $\text{Itin}_{x_0}(r)$.

There are two types of itineraries: finite and infinite. A point $\alpha \in \partial X$ has a finite itinerary if $\alpha \in \partial B$ for some block B and has an infinite itinerary otherwise. A boundary point having finite or infinite itinerary is independent of choice of basepoint (see [Moo10]). Let $\text{Itin}(\alpha) = B_1, \dots, B_n$ and let v_1, \dots, v_n be the vertices in \mathcal{T} associated to these blocks. The geodesic in \mathcal{T} between v_1 and v_n consists exactly of the vertices v_1, \dots, v_n . Furthermore, if $\text{Itin}(\alpha)$ is infinite, then the associated path is the vertices of a geodesic ray in \mathcal{T} starting at v_1 . From this we can define a map $p: \partial X \rightarrow \mathcal{T} \cup \partial \mathcal{T}$ which sends α to the last block in its itinerary or to the boundary point of $\partial \mathcal{T}$ associated to the itinerary of α .

We say that a geodesic ray is *lonely* if it is the only geodesic with its itinerary.

Finally we say that a subgroup is *finite itinerary* if every word in the subgroup has bounded itinerary. This is equivalent to saying that the subgroup stabilises a vertex of \mathcal{T} .

2.3. Right-angled Artin groups and Salvetti Complex. Throughout this section, let Γ be a graph.

Definition 2.7 (Right-angled Artin group). The *right-angled Artin group* with defining graph Γ , denoted A_{Γ} , is the group with presentation

$$A_{\Gamma} = \langle V(\Gamma) \mid [v_i, v_j] = 1 \iff \{v_i, v_j\} \in E(\Gamma) \rangle$$

Definition 2.8. Given a flag complex Γ we define X_{Γ} as follows:

For each vertex v_i in Γ , let $S_{v_i}^1 = S^1$ be a copy of the circle cubulated with 1 vertex. For each simplex $\sigma = [v_0, \dots, v_n]$ of Γ there is an associated torus

$T_\sigma = S_{v_0}^1 \times \cdots \times S_{v_n}^1$. If $\tau < \sigma$, then there is a natural inclusion $T_\tau \hookrightarrow T_\sigma$. Now define

$$X_\Gamma = \coprod_{\sigma < \Gamma} T_\sigma / \sim$$

where the equivalence relation \sim is generated by the inclusions $T_\tau \hookrightarrow T_\sigma$.

The *Salveti complex* S_Γ for A_Γ is the universal cover of X_Γ .

Remark 2.9. Some authors refer to X_Γ as the Salvetti complex and S_Γ as its universal cover. We are interested in S_Γ so shall refer to it as the Salvetti complex.

Proposition 2.10. *The 1-skeleton of the Salvetti complex for A_Γ is isomorphic to the Cayley graph of A_Γ .*

Definition 2.11. Let A_Γ be a RAAG we say that $H \leq A_\Gamma$ is a *parabolic subgroup*, if it is generated by a subset of the vertices of Γ .

Parabolic subgroups are convex in the sense that they stabilise a convex subcomplex of the Salvetti complex.

We will study the RAAG CK defined by a path with 4 vertices with the following presentation

$$CK = \langle a, b, c, d \mid [a, b], [b, c], [c, d] \rangle.$$

This group has a splitting as $(F_2 \times \mathbb{Z}) *_{\mathbb{Z}^2} (F_2 \times \mathbb{Z}) = \langle a, b, c \rangle_{\langle b, c \rangle} \langle b, c, d \rangle$. Whenever we refer to a splitting of CK throughout the paper this will be the splitting.

Lemma 2.12. *The Salvetti complex for CK has a connected block decomposition.*

Proof. The blocks are copies of $T_4 \times \mathbb{R}$ coming from the splitting $(F_2 \times \mathbb{Z}) *_{\mathbb{Z}^2} (F_2 \times \mathbb{Z})$. The union of the block boundaries is connected since each block boundary is the suspension of a Cantor set and they can be ordered in a way such that each intersects one of the previous block boundaries in S^1 . \square

3. BOUNDARIES OF AMALGAMATIONS

In this section we will study how the boundary of an amalgamated free product can be related to the boundaries of the component pieces.

Lemma 3.1. *Let X be a CAT(0) space and let A and B be closed subsets of X . Suppose there exists a closed subset C such that any geodesic from A to B passes through C . Then any path in the boundary between $\Lambda(A)$ and $\Lambda(B)$ passes through $\Lambda(C)$.*

Proof. Fix a basepoint $x_0 \in X$ and let $\gamma : [0, 1] \rightarrow \partial X$ be a path starting at $\alpha \in \partial A$ and ending at $\beta \in \partial B$. By the \mathcal{Z} -set property of \overline{X} , there is a homotopy $H : [0, 1] \times [0, 1] \rightarrow \overline{X}$ of γ such that $H(t, 1) = \gamma(t)$ and $H(t, s) \subset X$ for all $s \neq 1$. Furthermore, $H(0, s)$ is along the geodesic from

x_0 to α and $H(1, s)$ is along the geodesic from x_0 to β . Since C separates A and B , for sufficiently large s , any path from $H(0, s)$ to $H(1, s)$ passes through C . This gives a sequence of points in C which tend to infinity. Since $C \cup \Lambda(C)$ is compact, this sequence has a subsequence which converges in $\Lambda(C)$. By the construction, this point is along γ . \square

Using the above Lemma we can deduce our main theorem.

Theorem 3.2. *Let $G = A *_C B$ be a $CAT(0)$ group acting geometrically on a $CAT(0)$ space X . Suppose A and C act geometrically on subspaces X_A and X_C , respectively. Furthermore, suppose X_C and its translates separate X_A from the rest of X . Lastly, suppose X_A satisfies the following:*

- (1) X_A has a connected block decomposition,
- (2) ∂X_A is not path connected, and
- (3) $\Lambda(C) \subset \text{Nex}(X_A)$

then ∂X is not path connected.

Proof. Since $\text{Nex}(X_A)$ is path connected and ∂X_A is not path connected, we can find a point p in $\text{Nex}(X_A)$ and a point q in ∂X_A which cannot be connected by a path.

Suppose that we can find a path $\gamma: I \rightarrow \partial X$ from p to q . This path cannot be contained in ∂X_A as such γ passes through $\Lambda(C)$. This gives paths from p and q to $\Lambda(C)$. However $\Lambda(C)$ is contained in $\text{Nex}(X)$. Since $\text{Nex}(X)$ is path connected, we can then obtain a path from p to q in ∂X_A giving a contradiction. Thus there is no path from p to q in ∂X completing the proof. \square

3.1. Applications to right-angled Artin groups. We now move onto looking at some key examples coming from right angled Artin groups.

Proposition 3.3. *Let A_Γ be a RAAG such that $A_\Gamma = CK *_C B$ where C is a finite itinerary parabolic subgroup. Then ∂S_Γ is not path connected.*

Proof. Since C is a parabolic subgroup of A_Γ it acts on a convex subspace of S_Γ it also separated the space A_Γ . The Salvetti complex for CK has a connected block decomposition. This comes from the splitting of the form $(F_2 \times \mathbb{Z}) *_{\mathbb{Z}^2} (F_2 \times \mathbb{Z})$. The subspaces are copies of $T_4 \times \mathbb{R}$ and the walls are \mathbb{R}^2 . The block decomposition is connected since each block boundary is the suspension of a Cantor set and these block boundaries intersect along an S^1 .

Finally since C is a finite itinerary parabolic every ray in C ends in the boundary of a block and as such $\Lambda(C)$ is contained in the nexus of the block decomposition. We can now apply Theorem 3.2. \square

We obtain the following corollary immediately:

Corollary 3.4. *Let Γ be a path with at least 4 vertices. Then ∂S_Γ is not path connected.*

We can also do the case that the parabolics are not finite itinerary.

Lemma 3.5. *Let H be a proper parabolic subgroup of CK with an infinite itinerary ray α . Let β be an infinite itinerary ray in CK . Suppose β is not in H . Then α and β have different itineraries.*

Proof. There are 14 proper parabolic subgroups of CK . The only proper parabolic subgroups that contain infinite itinerary rays are $\langle a, b, d \rangle$, $\langle a, c, d \rangle$ and $\langle a, d \rangle$. By symmetry, we just have to check the cases of $\langle a, b, d \rangle$ and $\langle a, d \rangle$.

Let us first consider the case $H = \langle a, b, d \rangle$. By moving any b or d to the left we can assume that β has the form $w_1 c^n a w_2$. Where w_1 is a word in H . We see that the itinerary of β contains the vertex $v = w_1 c^n a \langle b, c, d \rangle$. We will show that this vertex is not on the itinerary of any ray in H .

Suppose that α is a ray with v in its itinerary. Then there is a finite prefix w of α such that $w \langle b, c, d \rangle = v \langle b, c, d \rangle$. Thus $w^{-1} w_1 c^n a \in \langle b, c, d \rangle$. However by the normal forms theorem for right angled Artin groups the a cannot be cancelled. Thus $v \langle b, c, d \rangle$ cannot be on the itinerary of α .

We now consider the case of $H = \langle a, d \rangle$. Since β is not in H β must contain at least one b or at least one c . We will study the case the β contains a b , the other case is the same. Since β is an infinite itinerary ray it contains infinitely many occurrences of d , as such there is at least one after the b . Thus β has the form $w_1 b w_2 d w_3$ for words w_i . Thus on the itinerary of this element is the vertex $w_1 b w_2 d \langle a, b, c \rangle$. We will show that this vertex cannot be on the itinerary of α .

Suppose that $w_1 b w_2 d \langle a, b, c \rangle$ is on the itinerary of α . This gives an element of h such that $h \langle a, b, c \rangle = w_1 b w_2 d \langle a, b, c \rangle$ or $h^{-1} w_1 b w_2 d \in \langle a, b, c \rangle$. Once again we see from the normal forms theorem for right angled Artin groups that this cannot be the case as there is no way to cancel the d to the right of the b . \square

Lemma 3.6. *Let α be an infinite itinerary ray contained in a proper parabolic subgroup H . Then α is lonely.*

Proof. We have already seen that α is separated from any ray not contained in H . However H is CAT(0) with isolated flats and the induced splitting from the splitting of CK respects the peripheral structure. Thus we can apply [Ben19, Lemma 10.17] to see that all infinite itinerary rays are lonely. \square

We can restate Proposition 6.22 from [Ben19] as follows:

Lemma 3.7. *Let α be an infinite itinerary ray contained in a proper parabolic subgroup. Then α is connected to the nexus of the block decomposition of CK .*

Theorem 3.8. *Let A_Γ be a RAAG which splits as $CK *_H B$ where H is a proper parabolic subgroup. Then ∂S_Γ is not path connected.*

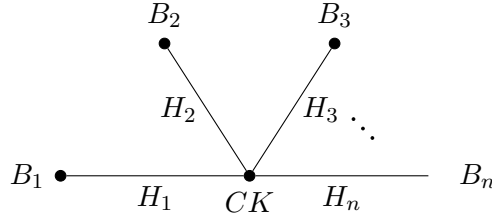


FIGURE 2. A graph of groups where each B_i is a RAAG and H_i is a proper parabolic subgroup of CK .

Proof. From the proof of Proposition 3.3 we can see that we have a subspace X_{CK} which has a connected block decomposition and boundary which is not path connected. Lemma 3.7 shows us that $\Lambda(H) \subset \text{Nex}(X_{CK})$ and so we can apply Theorem 3.2. \square

Corollary 3.9. *Let Γ be a graph isomorphic to a circle with at least five vertices. Then ∂S_Γ is path disconnected.*

Proof. Take any subpath of the circle of length 4. The two outside vertices of the path give a splitting as $A_\Gamma = CK *_{F_2} B$ where F_2 is the parabolic given by the two outside vertices. This is a proper parabolic subgroup of CK so we can apply Theorem 3.8. \square

It is worth noting that the idea of the previous proofs is that we start with two points which are not connected by a path in ∂CK and show that they are still not connected by a path in ∂G . Thus we can repeat the construction as long as we know that the two points in the ∂CK subgroup are not connected by a path in the previous stage. For instance we get the following:

Corollary 3.10. *Let A_Γ be a RAAG admitting a graph of groups as in Figure 2 where H_i is a proper parabolic subgroup of CK . Then ∂S_Γ is not path connected.*

In Figure 3, we draw attention to two graphs which are not joins and our methods cannot be applied.

We also study the groups from [Moo10]. These groups are of the form $(G_- \times \mathbb{Z}^n) *_{\mathbb{Z}^n} (\mathbb{Z}^n \times \mathbb{Z}^m) *_{\mathbb{Z}^m} (\mathbb{Z}^m \times G_+)$ where G_-, G_+ are infinite $\text{CAT}(0)$ groups. As discussed above these groups are shown to have non-unique $\text{CAT}(0)$ boundary [Moo10]. We obtain the following theorem about path connectedness.

Theorem 3.11. *Let G be of the form $(G_- \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times G_+)$ or $(G_- \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}^2) *_{\mathbb{Z}^2} (\mathbb{Z}^2 \times G_+)$, where G_- and G_+ are $\text{CAT}(0)$ groups. Then G acts on a $\text{CAT}(0)$ X and ∂X is not path connected.*

Proof. Let a be an infinite order element of G_- and d be an infinite order element of G_+ . Such elements exist by work of Swenson [Swe99].

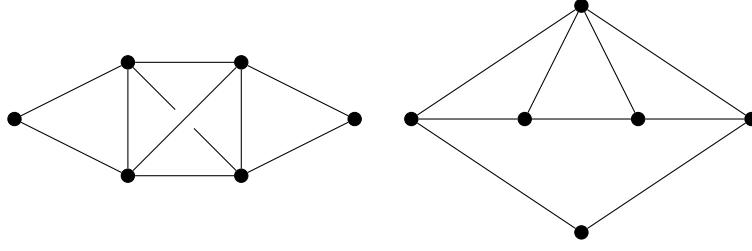


FIGURE 3. Two graphs for which we cannot apply our theorem and for which path connectedness of ∂S_Γ is unknown.

We obtain a splitting of $(G_- \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times G_+)$ as

$$(G_- \times \mathbb{Z}) *_{\mathbb{Z}^2} (CK) *_{\mathbb{Z}^2} (\mathbb{Z} \times G_+)$$

identifying $\langle a \rangle \times \mathbb{Z}$ with $\langle a, b \rangle$ and identifying $\mathbb{Z} \times \langle d \rangle$ with $\langle c, d \rangle$. We can build a CAT(0) space X on which this acts using the Equivariant Gluing Theorem [BH99, II.11.18]. As a subspace of this we have the Salvetti complex for CK and the amalgamating subgroups are finite itinerary and so the hypothesis of Theorem 3.2 are satisfied.

For $(G_- \times \mathbb{Z}) *_{\mathbb{Z}} (\mathbb{Z} \times \mathbb{Z}^2) *_{\mathbb{Z}^2} (\mathbb{Z}^2 \times G_+)$ we use the splitting

$$(G_- \times \mathbb{Z}) *_{\mathbb{Z}^2} (CK) *_{\mathbb{Z}^2 * \mathbb{Z}} (\mathbb{Z} \times (G_+ * \mathbb{Z})).$$

The proof from the first case follows verbatim for the second. \square

4. NON-UNIQUE BOUNDARIES

In this section, we prove Theorem 4.1, which we restate for convenience.

Theorem 4.1. *For each n , there is a group G_n and CAT(0) spaces X_n and Y_n admitting geometric group actions by G_n with the following properties:*

- ∂X_n and ∂Y_n are n -connected
- ∂X_n and ∂Y_n are not homeomorphic

It is known by [CK00] that the boundary of a CAT(0) group is not well defined. However, none of the known boundaries of this group are path connected. In this section we give examples of CAT(0) groups with non-unique boundary in which all boundaries are path connected. Moreover, we give similar results where all CAT(0) boundaries are n -connected for arbitrary n .

We follow closely the proof given in [CK00]. Let $X = X_\alpha$ be the universal cover of the Leeb complex where the geodesics in the central block meet at an angle of α . This space is a classifying space for CK . The groups that we will study are of the form $G_n = CK \times \mathbb{Z}^{n+1}$. This group acts properly cocompactly on $Z = X \times \mathbb{R}^{n+1}$. The boundary of $X \times \mathbb{R}^{n+1}$ is $\partial X * S^n$. We will show that we can still differentiate these spaces after taking a topological join with S^n . Note that this boundary is n -connected as ∂X is not empty and S^n is $n - 1$ connected.

The group G_n has a splitting as $F_2 \times \mathbb{Z}^{n+2} *_{\mathbb{Z}^{n+3}} F_2 \times \mathbb{Z}^{n+2}$. The space Z has a connected block decomposition where each block is isometric to $T_4 \times \mathbb{R}^{n+2}$ and the walls are each isometric to \mathbb{R}^{n+3} . Since each block B is isometric to the product of a tree and \mathbb{R}^{n+2} we see that $\partial B = \mathcal{C} * S^{n+1}$ where \mathcal{C} is a Cantor set. A *pole* of B is any point of S^{n+1} .

Lemma 4.2. [CK00, Lemma 3]

If B_1, B_2 are blocks, then one of the following holds:

- (1) $\partial B_1 \cap \partial B_2 = \emptyset$.
- (2) $B_1 \cap B_2$ is a wall W and $\partial B_1 \cap \partial B_2 = \partial W$.
- (3) There is a block B such that $B \cap B_i = W_i$ and $\partial B_1 \cap \partial B_2$ is the set of poles of B .

Proof. The blocks in our decomposition are a product of the blocks from [CK00] and \mathbb{R}^{n+1} . \square

Lemma 4.3. *Let z be a point of $X * Y = X \times Y \times [0, 1] / \sim$. Suppose $z = (x, y, t)$. Then a neighbourhood $N(z)$ of z in $X * Y$ is homeomorphic to*

$$N(z) = \begin{cases} N(x) \times C(Y) & \text{if } t = 0, \\ N(y) \times C(X) & \text{if } t = 1, \\ N(x) \times N(y) \times (t - \epsilon, t + \epsilon) & \text{if } 0 < t < 1. \end{cases}$$

Where $C(X)$ is the open cone on X .

Proof. The preimage of each the above sets under the quotient map is open thus they are open sets in the quotient. \square

Lemma 4.4. [CK00, Lemma 4]

Suppose that $\lambda \in \partial B$ and λ is not the pole of any block other than B . Then the path component of λ in a suitable neighbourhood Λ of λ is contained in ∂B .

Proof. Since the space in question is $\partial X * S^n$ we can understand neighbourhoods by 4.3. Since λ is not a pole of any other block we must be in the case $t \neq 1$. Thus the neighbourhood is homeomorphic to $N(x) \times D^{n+1}$ and the proof from [CK00] follows. \square

We say that λ is a *vertex* if there is a neighbourhood U of λ such that the path component of λ in U is homeomorphic to $D^n \times C(\mathcal{C})$. By the previous lemma, poles which do not come from S^n are vertices.

We say that a path is *safe* if it only goes through finitely many vertices and no points with a neighbourhood homeomorphic to $D^n \times C(\partial X)$. We say a path component is *safe* if there is a safe path between any pair of points in the path component. With this definition of safe path and vertex we recover the results of [CK00, Section 7].

Lemma 4.5. [CK00, Lemma 6]

The union of the block boundaries without S^n is a safe path component.

Lemma 4.6. [CK00, Lemma 7]

Let $c: [0, 1] \rightarrow \partial Z$ be a path and suppose that $c(0)$ has an infinite itinerary. Then either $c(t)$ has the same itinerary for all t or there is a point with finite itinerary.

Proof. If the path avoids the S^n in the decomposition $\partial Z = \partial X * S^n$, then the proof follows from [CK00, Lemma 7]. On the other hand every point on S^n has finite itinerary. \square

Corollary 4.7. [CK00, Corollary 8]

There is a unique safe component of ∂Z which is dense, namely the space describe in Lemma 4.5.

We say that a disk $D^{n+1} \subset \cup_B \partial B$ is a *simplex* if its of the form $I * S^n$ for an edge I defined in [CK00, Section 8]. As in [CK00] we see that simplices are contained in the boundary of a single block. Moreover, the boundary of a simplex is $S^0 * S^n$ and every point not in S^n is a vertex and no interior point of D^{n+1} is a vertex. We can see that all the boundary points not in S^n are either the poles of a single block or D^{n+1} is contained in ∂W for some wall W . We call an arc $I \subset \cup_B \partial B$ a *special edge* if every point is a vertex. Let x, y be vertices not on a special edge. Then they are in the same block if they are on the boundary of more than one simplex. They are in adjacent blocks if they are on the boundary of a unique simplex. Either both poles are on a special edge. A subset of $\cup_B \partial B$ is a block boundary if it is the union of all simplices intersecting it.

A *hemisphere* is a subset of ∂B of the form $S^n * I$ where I is a longitude from [CK00]. Let B be a block and \mathcal{P} be the set of poles in adjacent blocks. Let H be a hemisphere in B . Then by [CK00, Lemma 9], we have that $H \setminus \mathcal{P}$ has 3 components if $\alpha \neq \frac{\pi}{2}$ and 2 components if $\alpha = \frac{\pi}{2}$.

Finally, following [CK00, Section 10] we can distinguish these boundaries as any homeomorphism will take safe path components to safe path components, block boundaries to block boundaries, poles to poles and hemispheres to hemispheres. Then the previous paragraph gives a contradiction, concluding the proof of Theorem 4.1.

At this time there is no known example of a group one of whose boundaries is path connected and the other not. Moreover, there is no known example of a CAT(0) group admitting two non homeomorphic boundaries one of which is locally connected. Our examples do nothing to answer these questions.

REFERENCES

- [Ben19] Michael Ben-Zvi. Boundaries of groups with isolated flats are path connected. *arXiv e-prints*, page arXiv:1909.12360, Sep 2019.
- [Bes96] Mladen Bestvina. Local homology properties of boundaries of groups. *Michigan Math. J.*, 43(1):123–139, 1996.
- [BH99] Martin R. Bridson and André Haefliger. *Metric spaces of non-positive curvature*, volume 319 of *Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences]*. Springer-Verlag, Berlin, 1999.

- [BM91] Mladen Bestvina and Geoffrey Mess. The boundary of negatively curved groups. *J. Amer. Math. Soc.*, 4(3):469–481, 1991.
- [Bow98] Brian H. Bowditch. Cut points and canonical splittings of hyperbolic groups. *Acta Math.*, 180(2):145–186, 1998.
- [CK00] Christopher B. Croke and Bruce Kleiner. Spaces with nonpositive curvature and their ideal boundaries. *Topology*, 39(3):549–556, 2000.
- [CK02] C. B. Croke and B. Kleiner. The geodesic flow of a nonpositively curved graph manifold. *Geom. Funct. Anal.*, 12(3):479–545, 2002.
- [Gro87] M. Gromov. Hyperbolic groups. In *Essays in group theory*, volume 8 of *Math. Sci. Res. Inst. Publ.*, pages 75–263. Springer, New York, 1987.
- [Gui14] Craig R. Guilbault. Weak Z -structures for some classes of groups. *Algebr. Geom. Topol.*, 14(2):1123–1152, 2014.
- [Moo10] Christopher Mooney. Generalizing the Croke-Kleiner construction. *Topology Appl.*, 157(7):1168–1181, 2010.
- [Swa96] G. A. Swarup. On the cut point conjecture. *Electron. Res. Announc. Amer. Math. Soc.*, 2(2):98–100, 1996.
- [Swe99] Eric L. Swenson. A cut point theorem for $CAT(0)$ groups. *J. Differential Geom.*, 53(2):327–358, 1999.
- [Wil05] Julia M. Wilson. A $CAT(0)$ group with uncountably many distinct boundaries. *J. Group Theory*, 8(2):229–238, 2005.